A New Regularized Approach for Contour Morphing

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Abstract

In this paper, we propose a new approach for interpolating curves (contour morphing) in time, which is a process of gradually changing a source curve (known) through intermediate curves (unknown) into a target curve (known). The novelty of our approach is in the deployment of a new regularization term and the corresponding Euler equation. Our method is applicable to implicit curve representation and it establishes a relationship between curve interpolation and a two dimensional function. This is achieved by minimizing the supremum of the gradient, which leads to the Infinite Laplacian Equation (ILE). ILE is optimal in the sense that interpolated curves are equally distributed along their normal direction. We point out that the existing Distance Field Manipulation (DFM) methods are only an approximation to the proposed optimal solution and that the relationship between ILE and DFM is not local as it has been asserted before. The proposed interpolation can also be used to construct multiscale curve representation.

1 Introduction

Let C be a deformable closed planar curve such that $C(t), t \in [0,1]$ denotes a family of the evolved curves with known initial condition at $C(0) = C_0$ and $C(1) = C_1$. Our aim is to reconstruct a representation $C(t)$ for $0 < t < 1$ so that the sequence of intermediate shapes is smooth and continuous in time.

Curve interpolation has many applications in computer vision and graphics including a metric for comparing curves, animation, and higher resolution visualization from serial sections. This is different from interpolating function values at different grid points because we are reconstructing new contours from two known ones. It is also slightly different from the curve evolution problem because we have a fixed ending curve as well as starting curve.

Current solutions to curve interpolation are based on the Distance Field Manipulation (DFM) [18]. In DFM, the distance transformations of C_0 and C_1 are first represented as regular two dimensional functions. Then intermediate implicit functions are created by linear interpolation between the distance transformations. Finally, the curves are extracted as the zero-crossing of these intermediate functions [14, 15, 16, 18, 20, 21]. Although DFM has been widely used, it has not been sufficiently questioned. For example, 1) Why does the distance transformation work? 2) Are there alternative methods? 3) If there is an alternative method then is the distance transformation the optimal field function? This paper partially focuses on these questions by exploring an entirely new approach.

A unique feature of our method is in the new regularization term and the corresponding Euler-Lagrange equation. Since interpolation is highly under-constrained, a good guess about the transformation process is equivalent to finding a function $f(x,y)$ defined in a certain region R such that $f(C_0) = 0, f(C_1) = 1$ and $C(t)$ is the level curve defined by $f(x,y) = t$. This function can then be approximated by a regularization approach that is well known in computer vision. Examples include shape from shading [19], surface reconstruction [4], reconstruction from projections [6], *etc.* Current regularization techniques require that the surface be globally smooth, and the most popular form is to minimize $\int_R g(f)$. Let $\bigtriangledown$ be the gradient operator and $\triangle$ the Laplacian operator, g could be $g = |\bigtriangledown f|$, $g = |\bigtriangledown f|^2$, $g = |\triangle f|$, or other variations. These techniques have no control on local properties of f since minimization is applied to the integral, and f might change sharply. Here, we propose a new regularization term based on minimizing the supremum of $|\bigtriangledown f|$. We prove that the Euler equation of this functional is the Infinite Laplacian Equation (ILE). We also show that the solution of ILE is *optimal* for the interpolation problem since $C(t)$ will be equally distributed along the gradient trajectory. We also show that (1) while DFM is efficient and simple, it is only an approximate solution to ILE; (2) the solution based on DFM is a global result, and it is not a widely accepted local operation [21]; and (3) the interpolation technique can be used for multiscale curve representation.

Section 2 provides a summary of contour representation and proposed new regularization. Section 3 outlines the corresponding numerical solution and compares the behavior of the energy function to the

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standard gradient based regularization. Section 4 provides experimental results and its application to multiscale contour representation. Section 5 concludes the paper.

2 Interpolation with Infinite Laplacian Equation

Let O_i be the inside-outside function of curve C_i, $i = 0, 1$,

$$O_i(x,y) = \begin{cases} -1, & \text{if } (x,y) \text{ is inside } C_i \\ 1, & \text{if } (x,y) \text{ is outside } C_i \\ 0, & \text{if } (x,y) \text{ is on } C_i \end{cases} \quad (1)$$

We then define the "Interpolation Region" R as

$$R(C_0, C_1) = \{(x,y) | O_0(x,y) O_1(x,y) \leq 0\} \quad (2)$$

Examples of R are shown in Figure 1.

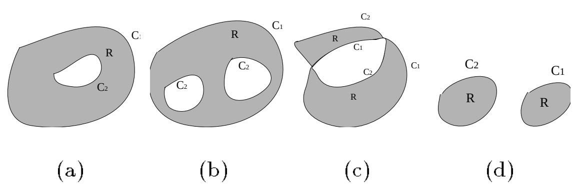


Figure 1: $R(C_0, C_1)$: (a) Doubly connected region, (b) Multiply connected region, (c) Intersection, and (d) Isolated objects.

We restrict the interpolation to R, *i.e.*, $C(t) \subset R$. As t changes from 0 to 1, $C(t)$ changes from C_0 to C_1 continuously and smoothly, and sweeps every point in R. Suppose $f(x,y)$ be the time at which the curve crosses a given point (x,y). Then function f satisfies

$$C(t) = \{(x,y) | f(x,y) = t\} \quad (3)$$

Equation (3) gives a implicit representation of the interpolated curves. Suppose that u is the parameter of the curve. Differentiating both sides of $f(x,y) = t$ with respect to u, we have

$$\bigtriangledown f \cdot \frac{\partial C}{\partial u} = 0 \quad (4)$$

Where $\frac{\partial C}{\partial u}$ is in the tangent direction, and $\bigtriangledown f$ is in the normal direction of the curve respectively. Now, consider the evolution of the curve, where any deformation [7, 11, 12, 13] can be written as:

$$\frac{\partial C}{\partial t} = \beta N + \alpha T \quad (5)$$

where N is the normal vector and T the tangent vector. β and α are arbitrary functions. It is well known that the tangent component has no effect on the shape of the deformed curve, but only changes the parameterization[11, 12, 13]. We can then set $\alpha = 0$. Thus, the curve moves along the normal direction with the speed $\beta(u,t)$. In this context, differentiating $f(x,y) = t$ with respect to t yields

$$\bigtriangledown f \cdot \frac{\partial C}{\partial t} = 1 \quad (6)$$

Since $\bigtriangledown f$ and N are in the same direction,

$$|\bigtriangledown f| \beta = 1 \quad (7)$$

We have,

$$\beta = \frac{1}{|\bigtriangledown f|} \quad (8)$$

Thus, the gradient of f contains both the direction and speed information of the curve movement.

2.1 A new regularization term

To interpolate the curves implicitly, we need to solve the following problem

Find $f(x,y), (x,y) \in R$, such that
$f(C_0) = 0, f(C_1) = 1$.

Once f is found, $C(t)$ can be obtained by Equation (3). This problem is still under-constrained. However, since we changed the problem from curve interpolation to functional interpolation, well known mathematical tools, such as regularization, can be leveraged. Most of the existing regularization techniques attempt to minimize an integral such as $\int_R |\bigtriangledown f|^2$. Yet, this formulation has no control on the local property of f. In other words, the global average of $|\bigtriangledown f|$ may be small, but locally f may change sharply. A better way to overcome this issue is to minimize $|\bigtriangledown f|$ at every point. Thus, we formulate the problem as the minimization of the supremum of $|\bigtriangledown f|$. It is natural to consider the functional $H(f) = sup_R |\bigtriangledown f|$ as a "limit" of the sequence of functionals

$$H_N(f) = (\int_R |\bigtriangledown f|^{2N} dx)^{\frac{1}{2N}}, \quad N = 1, 2, 3, ... \quad (9)$$

The Euler equation for the minimization of the functional $H_N(f)$ can be expressed as:

$$|\bigtriangledown f|^{2(N-2)} \{ \frac{1}{2(N-1)} |\bigtriangledown f|^2 \bigtriangleup f + f_x^2 f_{xx} + 2 f_x f_y f_{xy} + f_y^2 f_{yy} \} = 0 \quad (10)$$

where a subscript indicates a derivative, such as $f_x = \frac{\partial f}{\partial x}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$. By removing the first coefficient and letting $N \rightarrow \infty$, we have

$$f_x^2 f_{xx} + 2 f_x f_y f_{xy} + f_y^2 f_{yy} = 0 \quad (11)$$

Equation (11) is called the Infinite Laplacian Equation(ILE), which has been studied widely in the literature [1, 2, 3, 10]. Some important properties of the equation [2] are: (1) There is at most one solution (that may not be a smoot curve) and if we redefine the "solution" in a suitable weak sense, then a solution continuous at least to C^1 does exist; (2) The trajectory of the gradient of f is either a convex curve or a straight line; and (3) There are no stationary points $|\bigtriangledown f| = 0$ in R.

Introducing the notation $\mathcal{J}(f) = f_x^2 f_{xx} + 2f_x f_y f_{xy} + f_y^2 f_{yy}$, our problem now becomes

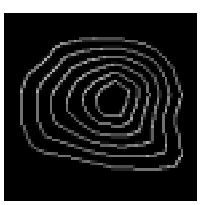
$$\textit{Find } f(x,y), (x,y) \in R, \textit{ such that} \\ f(C_0) = 0, f(C_1) = 1, \textit{and } \mathcal{J}(f) = 0$$

2.2 Equal Importance Criteria

This section outlines the rationale for optimality of the supremum as the norm for regularization. Our argument is based on the *Equal Importance Criterion* [8, 9]. This criterion asserts that *every point in R is equally important and contributes similarly to the reconstruction process. Any other assumption means that we need to know some additional information about the curve.* It is easy to verify that ILE is equivalent to the following equation

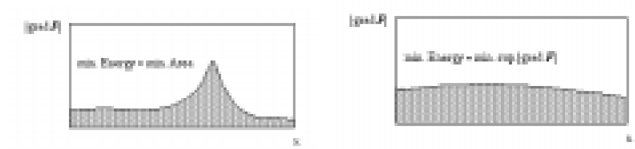
$$\mathcal{J}'(f) = \bigtriangledown(|\bigtriangledown f|) \cdot \frac{\bigtriangledown f}{|\bigtriangledown f|} = 0 \qquad (12)$$

which implies that along each trajectory of the gradient of f, the magnitude of the gradient is a constant. The interpolated curves $C(u,t)$ are then equally distributed along their normal direction, or simply each point advances at its own constant speed, as shown in Figure 2. When no information about the deformation process is available, it is best to assume that the curves $C(u,t)$ are equally distributed since if two curves are closer to each other than to other curves, there must be some additional reason for such variation that should be known. This is in conflict with the problem definition. Thus, in view of time or distance between curves, which is our only clue about the curve, all points are equally important. Comparative analysis of our approach indicates that our method generates a more smooth family of curves. This is shown in Figure 3, which indicates that minimizing the integral of $|\bigtriangledown f|$ aims to minimize the area while minimizing the supremum aims to minimize the maximum. Although the overall integral of $|\bigtriangledown f|$ may be larger, the supremum is smaller in our approach and the gradient is more likely to concentrate at a smaller range. Thus, speed of the moving curve is in a smaller range and the whole curve changes more smoothly with time.



(a)

Figure 2: Curves are equally distributed along the gradient trajectory. The most outer and inner curves are labeled as known source and target curves. Other curves are interpolated ones.



(a) (b)

Figure 3: Comparison of our method with traditional regularization terms. (a) Solution to minimization of gradient; (b) Solution to minimization of supremum.

3 Numerical Solution of ILE

Many numerical methods can be used to solve Equation (11), and at least a weak solution is guaranteed. In this section, we first give an iterative method. Then we show that a simple approximate solution can be constructed by DFM.

3.1 The iterative approach

The solution of ILE can be obtained iteratively by:

$$f^{v+1} = f^v - \delta \mathcal{J}(f^v) \qquad (13)$$

where δ is the step size and v indicates the iteration number. Although the approach converges only to a local minimum, the solution is acceptable if we start from a good initialization. The derivatives can be calculated by finite difference:

$$\begin{aligned} f_x(i,j) &= f(i,j) - f(i-1,j) \\ f_y(i,j) &= f(i,j) - f(i,j-1) \\ f_{xx}(i,j) &= f(i-1,j) + f(i+1,j) - 2f(i,j) \\ f_{yy}(i,j) &= f(i,j-1) + f(i,j+1) - 2f(i,j) \\ f_{xy}(i,j) &= f(i-1,j-1) + f(i,j) \\ &\quad -f(i-1,j) - f(i,j-1) \end{aligned} \qquad (14)$$

The Algorithm can be written as:

Algorithm 1 Iterative Solution

1. Initialize f, with the boundary condition $f(C_0) = 0, f(C_1) = 1$.
2. Initialize R with the distance transform.

3. Update all the points inside R with equation (13).

4. Compute $sup|\bigtriangledown f|$.

5. Repeat 2 to 3 until a local minimum of $sup|\bigtriangledown f|$ is reached.

6. Find the interpolated curves $C(t) = \{(x, y)|f(x, y) = t\}$.

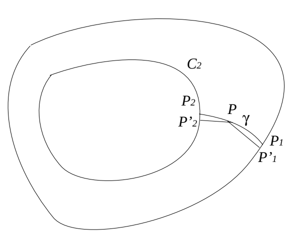
Initialization of R can be calculated by DFM since it is a good approximation of the solution (This will be shown in the next section).

3.2 Interpolation with Distance Transform

The iterative solution for optimality can be costly especially if the topologies of C_0 and C_1 are complicated. However, leveraging the distance transform can yield a more efficient solution.

Let's define $\mathcal{T}_i$ as the *Signed Distance Transformation* of $C_i, i = 0, 1$, where $\mathcal{T}_i(x, y)$ is the distance from (x, y) to the nearest point on C_i, and the distance is set to a negative number if (x, y) is inside C_i and positive otherwise. For each point p (shown in Figure 4), there should be a gradient trajectory γ passing through it such that it intersects C_0 and C_1 at p_0 and p_1, respectively. Since the normal of these two curves and the gradient of f are in the same direction, $\gamma \perp C_0$ at p_0 and $\gamma \perp C_1$ at p_1. We can approximate the curve γ, passing through p, by drawing two line segments $pp_0' \perp C_0, pp_1' \perp C_1$, to create $p_0'pp_1'$. Let l denote the length of γ from p_0 to p_1. Hence, $l \approx |p_0'p| + |p_1'p|$. The preceding formulation indicates that $|p_0'p| = -\mathcal{T}_0(p)$, $|p_1'p| = \mathcal{T}_1(p)$. Since f is changing linearly from 0 to 1 along γ, $f(p)$ can be approximated by:

$$f(p) = \frac{|p_0'p|}{|p_0'p| + |p_1'p|} = \frac{-\mathcal{T}_0(p)}{\mathcal{T}_1(p) - \mathcal{T}_0(p)} \tag{15}$$



(a)

Figure 4: Computing $f(x, y)$ from distance transformation

Equation (15) has a drawback that, when C_0 and C_1 intersect each other at p, we get $\mathcal{T}_1(p) - \mathcal{T}_0(p) = 0$ and a zero divisor. Thus, we seek an isocurve representation without this defect. The two dimensional isocurve representation, $\phi(x, y, t), 0 \leq t \leq 1$, can be expressed as:

$$\phi(x, y, t) = t\mathcal{T}_1(x, y) + (1 - t)\mathcal{T}_0(x, y) \tag{16}$$

Note that this is exactly the DFM. Isocurve $\phi(x, y, t) = 0$ locates at

$$t(x, y) = \frac{-\mathcal{T}_0(x, y)}{\mathcal{T}_1(x, y) - \mathcal{T}_0(x, y)} \tag{17}$$

which is exactly the curve that we reconstructed in (15). Thus, DFM is an approximation of the ILE solution in the sense described above.

Equation (16) is preferred over equation (15) because it works for any C_0 and C_1 even if $C_0 = C_1$. Thus, the method treats any curve and topological changes naturally and cannot fail. Let $0 \leq t \leq 1$ in equation (17) then $\mathcal{T}_1(x, y)\mathcal{T}_0(x, y) \leq 0$. We know that $\mathcal{T}_1(x, y)\mathcal{T}_0(x, y) \leq 0$ in and only in $\mathcal{R}$. Thus, the isocurve $\phi(x, y, t) = 0$ is guaranteed to be inside the region R, $C(t) \subset R, 0 \leq t \leq 1$. The new algorithm is as follows:

Algorithm 2 Algebraic Solution

1. Calculate the $\mathcal{T}_i, i = 0, 1$.

2. for t = 0 to 1

   (a) Calculate the isocurve representation $\phi(t) = t\mathcal{T}_1 + (1 - t)\mathcal{T}_0$.

   (b) $C(t)$ equals the zero-crossing of $\phi(t)$.

The DFM approach has following advantages:

1. For arbitrary C_0, C_1, we can get a smooth and natural interpolation.

2. The interpolation can be carried out at any desired resolution.

3. The isocurve representation is good for geometric analysis.

4 Applications and Experimental Results

In this section, we apply the proposed technique to different curves and show that a new multiscale curve representation can be constructed. In every experiment, the first curve is the source curve and the last curve is the target curve. Others are interpolated intermediate curves.

Figures 5 and 6 show the results computed by the variational approach where f is initialized by DFM (Algorithm 2). We then update the implicit function until a local minimum is reached. Finally, the level curves at different height are extracted.

Figure 5 shows interpolation of a fish shape to a panda. The image size is 162 by 161 where 8 curves have been inserted. In this experiment, $\delta = 0.05$.

Figure 6 shows interpolation of a vase to a cat. The image size is 163 by 183 and 8 curves have been inserted. In this experiment, $\delta = 0.03$.

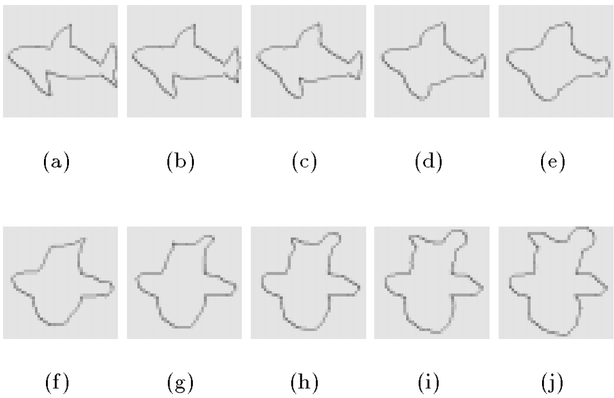


Figure 5: Interpolating fish and panda.

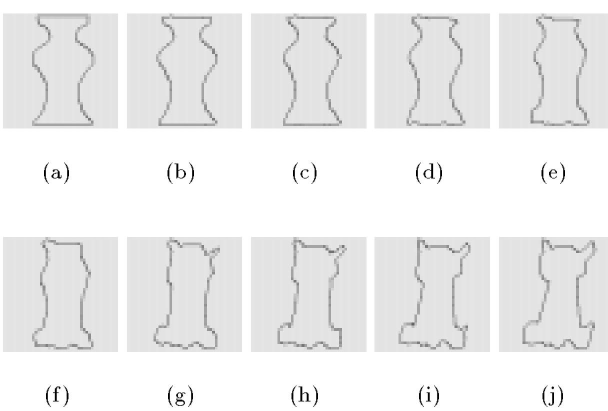


Figure 6: Interpolating vase and cat.

Figures 7 and 8 show interpolated results by DFM for complex topological changes. The distance transformation is calculated by the methods proposed by Borgefors [5]. Figure 7 shows interpolation of character "F' to "O". The image size is 120 by 121. Eight images have been interpolated between times 0 and 1.

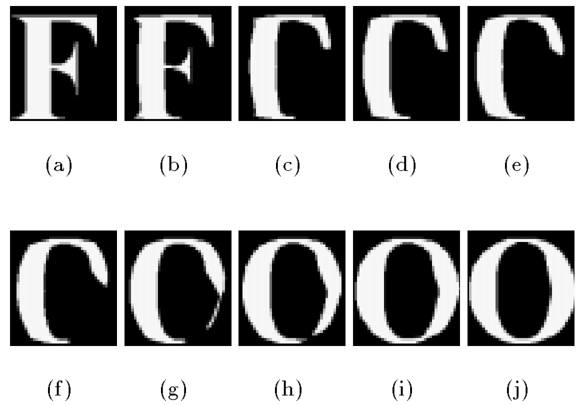


Figure 7: Interpolating character "F" and "O"

Figure 8 shows interpolation of character "B" to "L". The image size is 104 by 111. Eight images have been inserted between times 0 and 1.

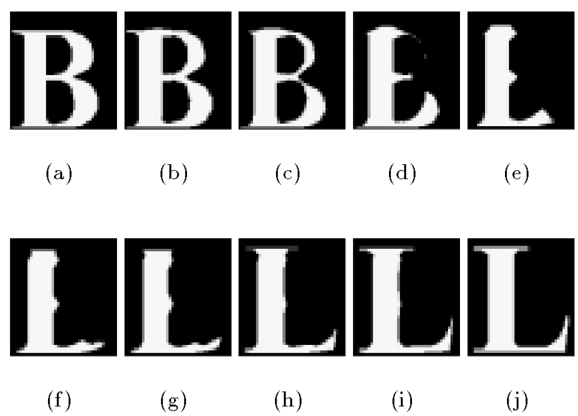


Figure 8: Interpolating character "B" and "L"

4.1 Multiscale curve representation

Traditional multiscale curve representation used Gaussian smoothing [17] or curvature deformation [8, 11, 12, 13], and it was proved that the curve converges to a circle [11, 12]. Saprio also constructed the length and area preserving deformation. The proposed curve interpolation can be used for multiscale curve representation with C_1 as the original curve and

C_2 as a circle with desired radius. Figures 9 and 10 show the experimental results. Note that the reconstructed curves are dependent on the relative position of circle and its radius. In all experiments, the circle is centered at the mass center of the curve and the radius is one fourth of the length of the curve outside box. All curves are computed by the iterative approach of algorithm 1.

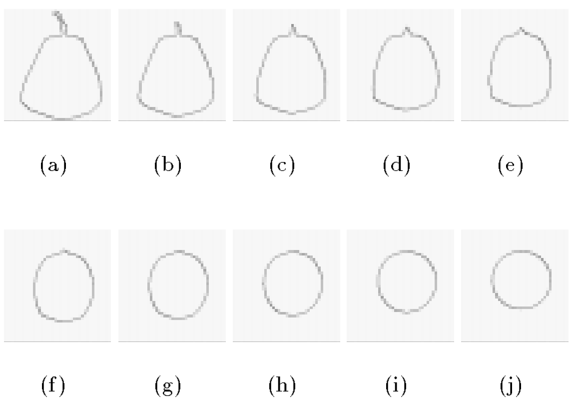


Figure 9: Multiscale representation of pear shape

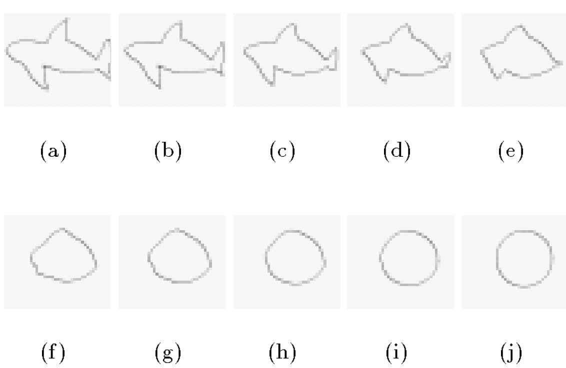


Figure 10: Multiscale representation of fish shape

5 Conclusion

We proposed a PDE based solution to the contour morphing problem. Our method finds natural and smooth interpolated curves that are equally distributed along the normal direction. This is optimal when no information about the deformation process exists and the best thing that we can do is to assign the generated curves fairly. The PDE is derived from a new regularization term that ensures the local smoothness. A numerical method was developed to construct and compute an optimal solution. At the same time, we showed that DFM is an efficient and simple approximation to the ILE, which can handle any curve with arbitrary topological changes. The method is applicable to problems in computer vision and graphics.

References

[1] G. Aronsson. Extension of function satisfying lipschitz conditions. *Arikv för Matematik*, 6:551–561, 1966.

[2] G. Aronsson. On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$. *Arikv för Matematik*, 7:133–151, 1967.

[3] G. Aronsson. On certain singular solutions of the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$. *Manuscripta Math*, 41:133–151, 1981.

[4] R.M. Bolle and B.C. Vemuri. On three-dimensional surface reconstruction methods. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13(1):1–13, January 1991.

[5] G. Borgefors. Distance transformations in arbitrary dimensions. *Computer Vision, Graphics, and Image Processing*, 27:321–345, 1984.

[6] Y. Bresler, J. Fessler, and A. Macovski. A bayesian approach to reconstruction from incomplete projections of a multiple object 3d domain. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 11(8):840–858, 1988.

[7] G. Cong and S. Ma. Corner enhancement in curvature scale. *Pattern Recognition*, 31(10):1491–1501, 1998.

[8] G. Cong and B. Parvin. Shape from equal thickness contour. In *Proceedings of the Conference on Computer Vision and Pattern Recognition*, pages 502–507, 1998.

[9] G. Cong and B. Parvin. An algebraic solution to surface recovery from cross-sectional contours. *Graphical Models and Image Processing*, 61(4):222–243, 1999.

[10] L. Evans. Estimates for smooth absolutely minimizing lipschitz extensions. *Electronic Journal of Differential Equations, http://ejde.math.swt.edu/Volumes/1993/03-Evans/abstr.html*, 1993(3):1–9, 1993.

[11] M. Gage and R. Hamilton. The heat equation shrinking convex plane curves. *Journal of Differential Geometry*, 23:69–96, 1986.

[12] M. Grayson. The heat equation shrinks embedded plane curve to round points. *Journal of Differential Geometry*, 26:285–314, 1987.

[13] B. Kimia, A. Tannenbaum, and S. Zucker. On the evolution of curves via a function of curvature, i: The classic case. *JMAA*, 163(2), 1992.

[14] A. Lee, D. Dobkin, W. Sweldens, and P. Schroder. Multiresolution mesh morphing. In *Proceedings of the SIGGRAPH Conference*, pages 343–350, 1999.

[15] D. Levin. Multidimensional reconstruction by set-valued approximation. *IMA J. Numerical Analysis*, 6:173–184, 1986.

[16] D. Levin and A. Solomovici. Three-dimensional distance field metamorphosis. *ACM Transactions on Graphics*, 17(2):116–141, 1998.

[17] F. Mokhatarian and A. Mackworth. A theory of multiscale curvature based shape representation for planar curves. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 14:789–805, 1992.

[18] B. Payne and Arthur Toga. Distance field manipulation of surface models. *IEEE Computer Graphics and Applications*, 12(1):65–71, 1992.

[19] A. Pentland. Linear shape from shading. *International Journal of Computer Vision*, 4:153–162, 1990.

[20] S. Raya and J. Udupa. Shape-based interpolation of multidimensional objects. *IEEE Transactions on Medical Imaging*, 9(1):32–42, 1990.

[21] G. Turk and J. O'Brien. Shape transformation using variational implicit functions. In *Proceedings of the SIGGRAPH Conference*, pages 335–342, 1999.